Wave formation in laminar flow down an inclined plane

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SUMMARY

This paper deals theoretically with a problem of hydrodynamic stability characterized by small values of the Reynolds number R. The primary flow whose stability is examined consists of a uniform laminar stream of viscous liquid running down an inclined plane under the action of gravity, being bounded on one side by a free surface influenced by surface tension. The problem thus has a direct bearing on the properties of thin liquid films such as have important uses in chemical engineering.

Numerous experiments in the past have shown that in flow down a wall the stream is noticeably agitated by waves except when R is quite small; on a vertical water film, for instance, waves may be observed until R is reduced to some value rather less than 10. The present treatment is accordingly based on methods of approximation suited to fairly low values of R, and thereby avoids the severe mathematical difficulties usual in stability problems at high R. The formulation of the problem resembles that given by Yih (1954); but the method of solution differs from his, and the respective results are in conflict. In particular, there is disagreement over the matter of the stability of a strictly vertical stream at very small R. In contrast with the previous conclusions, it is shown here that the flow is always unstable: that is, a class of undamped waves exists for all finite values of R. However, the rates of amplification of unstable waves are shown to become very small when R is made fairly small, and their wavelengths to become very large; this provides a satisfactory explanation for the apparent absence of waves in some experimental observations, and also for the wide scatter among existing estimates of the 'quasi-critical' value of R below which waves are undetectable. In view of the controversial nature of these results, emphasis is given to various points of agreement between the present work and the established theory of roll waves; the latter theory gives a clear picture of the physical mechanism of wave formation on gravitational flows, and in its light the results obtained here appear entirely reasonable.

The conditions governing neutral stability are worked out to the third order in a parameter which is shown to be small; but a less accurate approximation is then justified as an adequate basis for an easily workable theory providing a ready check with experiment, This theory is used to predict the value of R at which observable waves should first develop on a vertical water film, and also the length and velocity of the waves. These three predictions are compared with the experimental results found by Binnie (1957), and are substantially confirmed.

1. Introduction

The flow of liquid in a thin film down a solid wall may often be observed in everyday life, as when rain water runs in a sheet down a window pane, or when paint, or mayonnaise, drains from some solid object which has been dipped in the liquid. It is also a subject of practical importance in chemical engineering, and has been studied by many experimental workers concerned with that field (e.g. Kirkbride 1934; Friedman & Miller 1941; Grimley 1945; Dukler & Bergelin 1952). The character of the flow has been shown to depend largely on the Reynolds number $R = Q/\nu$, where Q is the rate of volume flow per unit span of the stream and ν the kinematic viscosity ($=$ viscosity μ/density ρ), although the influence of surface tension clearly may be important in most of the cases studied. In flow down a vertical plane, for example, the motion is apparently turbulent when R is greater than about 300 (Jeffreys 1925). When R is less than this, the mean flow is evidently governed by a law of laminar friction, and the mean depth is found to be approximately that given by the simple theory due to Nusselt (1916) and Jeffreys (1925) which assumes a uniform flow. Nevertheless, waves are to be observed on the free surface throughout almost the entire range of laminar flow, the possible exception being when R is reduced below a certain small value which may be about 4 if the liquid is water.

The apparent absence of waves on very thin films has led several investigators to assume that, for the flow down a vertical plane, there exists a critical value of R below which uniform laminar flow is entirely stable: that is, the flow is in a condition where small disturbances of every kind are suppressed. The well-proven significance of the Reynolds number in its more familiar contexts has undoubtedly encouraged this belief. Indeed, numerous estimates of the supposed critical Reynolds number have been stated (see Binnie (1957) for a review of some of them). On the theoretical side the work of Kapitza (1948) led to an estimate of 5·8 for the critical value, and that of Yih (1954) gave about 1·5.

In the present paper it will be argued that a critical Reynolds number in the usual sense does *not* exist for the particular case of uniform flow down a vertical plane. In other words, for all finite Reynolds numbers there is a class of wave-like disturbances which undergo unbounded amplification according to a linearized theory. The presence or absence of surface tension does not alter this general conclusion. The theory is, however, far from being incompatible with the various experiments seeming to indicate a critical Reynolds number, since an alternative explanation is readily forthcoming for the absence of observable waves. In fact the new explanation demonstrates a unity among various experimental results where the respective values of the supposed critical Reynolds number are conflicting. On the other hand, it is necessary to dispute previous theoretical work on this subject.

The problem to be considered could be adapted to a simplified treatment akin to the usual theory of 'roll waves' on turbulent streams. In the latter only the 'overall' features of the flow are considered: that is, the frictional force against the stream is taken to vary in some prescribed way with the local depth and mean velocity and to be independent of the finer details of the motion, while the action of gravity (whose component perpendicular to the flow tends to keep the free surface flat) is expressed in a momentum equation based on the mean velocity. In his theory of roll-wave formation, Dressler (1952) derived a stability condition for a generalized law of friction, which extends to the example of laminar flow in question here. Thus, as the terms representing the restoring forces can easily be modified to encompass surface tension as well as gravity, immediate use may be made of his result to estimate when waves should appear. In § 5 of this paper the result following from Dressler's theory is shown to correspond to a limiting case of the present theory; but it is appropriate to note here that an approach to the problem directly by way of Dressler's work effectively demonstrates the instability inherent at all Reynolds numbers when the plane is vertical. This is particularly encouraging since roll-wave theory keeps a much closer connection with physical reasoning than is possible in the present treatment.

Since the primary (waveless) flow is laminar and completely known, the stability problem can be formulated mathematically in a more precise way than that of roll-wave theory. The formulation is made in the well-known manner which has been extensively used for stability problems concerned with laminar flows between fixed boundaries (cf. Lin 1955). It is implicitly assumed that the Fourier components of an arbitrary small disturbance are dynamically independent; thus the solution for a wave of arbitrary period and velocity constitutes a complete solution. The equations of motion for a wave of small amplitude are then set up, together with appropriate boundary conditions; this defines a characteristic-value problem leading to a relationship among the essential parameters of the primary flow and the wave. There is particular interest in the conditions under which the wave is propagated unchanged (i.e. its velocity is a real constant), since these refer to the dividing line between stability and instability. These few sentences merely outline a very well-known type of problem, and we may refer to the book by Lin (1955) for an extensive account of the subject. The novelties of the present application are, first, the introduction of boundary conditions appropriate to a free surface under the action of gravity and surface tension, and, second, the use of a method of approximate solution suited to fairly small Reynolds numbers. The latter avoids the serious difficulties which beset many of the well-known problems of hydrodynamic stability, in which instability is expected only for large Reynolds numbers (Lin 1955, p. 7). Furthermore, in a later part of this paper justification is found for throwing off most of the weight of an analysis along the usual lines of stability theory, and hence proceeding with an easily workable approximation which is able to give a satisfactory account of all the main features of wave formation in practice. A rather similar formulation of this problem has already been given by Yih (1954), except that no account was taken there of surface tension; however, since the present method of solution is different and the results entirely so, there seems to be sufficient reason for setting out the argument completely from the beginning.

The analysis will be restricted to two-dimensional wave disturbances in a vertical plane (x, y). This simplification may be justified by appeal to the well-known argument first given by Squire (1933; restated by Lin 1955, § 3.1) in connection with the problem of disturbed flow between parallel planes. He pointed out that any three-dimensional disturbance is governed by the same equations as a certain two-dimensional disturbance in a similar flow at lower Reynolds number. This means that two-dimensional waves have a greater tendency to instability than three-dimensional ones; and it follows that a two-dimensional analysis is completely adequate if only the stability of a flow is in question. The suitability of a two-dimensional analysis is further demonstrated by Binnie's experiments, noted immediately below, which showed that waves at fairly small Reynolds numbers are very approximately uniform along the horizontal line of their crests.

The experimental results reported by Binnie (1957) have been found to provide confirmation of the theory in several important respects. These results were kindly made known to me during the course of my theoretical work, and I was encouraged by them to explore the long-wave approximation described in § 5. The experiments were made with vertical water films, and their results bear out the theoretical predictions on the following three counts: (i) the Reynolds number at which large amplifications of unstable disturbances first occur, so that observable waves develop; (ii) the velocity of the wave of maximum instability; (iii) the corresponding wavelength. The theoretical argument also accounts for the irregularity of the observed waves.

2. Properties of the primary flow

The laminar flow whose stability is to be examined is a uniform two-dimensional stream bounded on one side by a fixed wall and on the other by a free surface; the fluid has constant density and viscosity. The velocity U is everywhere parallel to x (see figure 1), and the graph of U *vs* y is a parabola which has its vertex in the free surface since, as the fluid is viscous, the rate of shearing dU/dy must vanish there. In this paper the coordinates (x, y) are made non-dimensional by taking the stream depth h as the unit of length, so that $y = 0$ and $y = 1$ define the free surface and the fixed boundary respectively. Also, velocities are made non-dimensional by taking the undisturbed velocity at $y = 0$, denoted by u_0, as the unit of velocity.

Thus we have $$U = 1-y^2. \tag{2.1}$$

By integration this immediately shows that the dimensionless mean velocity is $u_m = \frac{2}{3}$ (i.e. in dimensional units $u_m = Q/h = \frac{2}{3}u_0$). The motion is steady, so that the shear force $-(\mu u_0/h)\,dU/dy$ on the wall is balanced by the total gravity force on the stream in the direction of flow; thus

$$2\mu u_0/h = \rho hg\sin\theta,$$

where θ is the slope (figure 1). This gives directly

$$u_0 = \tfrac{1}{2}h^2 g\sin\theta/\nu, \tag{2.2}$$

$$R = Q/\nu = \tfrac{2}{3}u_0 h/\nu = \tfrac{1}{3}h^3 g\sin\theta/\nu^2. \tag{2.3}$$

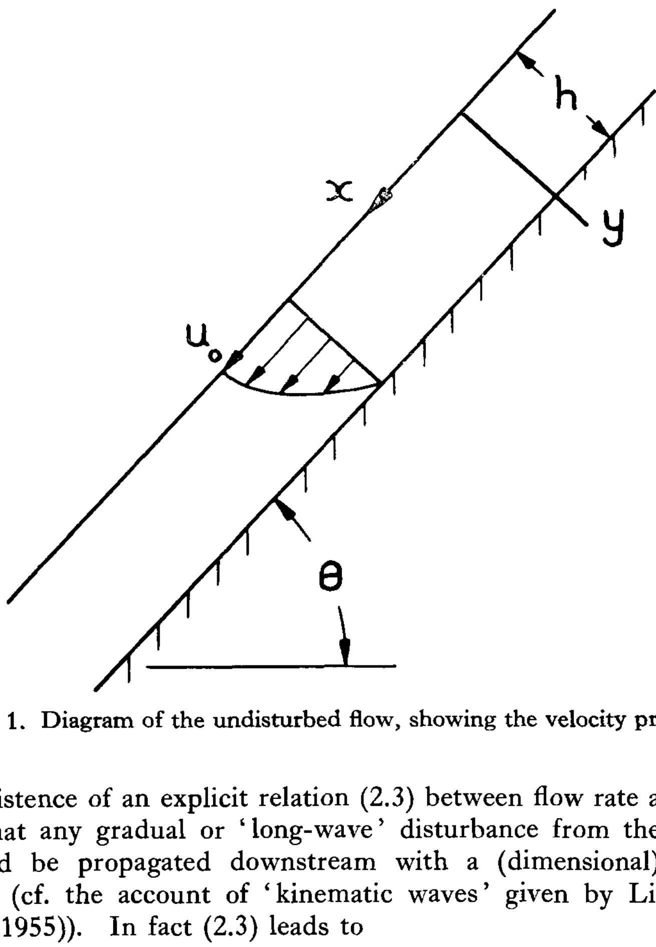

Figure 1. Diagram of the undisturbed flow, showing the velocity profile.

The existence of an explicit relation (2.3) between flow rate and depth suggests that any gradual or 'long-wave' disturbance from the uniform state would be propagated downstream with a (dimensional) velocity $C = dQ/dh$ (cf. the account of 'kinematic waves' given by Lighthill & Whitham (1955)). In fact (2.3) leads to

$$C = h^2 g\sin\theta/\nu = 2u_0. \tag{2.4}$$

This simple result provides an interesting check on the subsequent analysis, which shows by a totally different method that very long periodic waves are indeed propagated with a phase velocity equal to $2u_0$.

3. The stability problem

Suppose that the free surface is given a perturbation represented by the equation

$$y = \eta(x,t) = \delta e^{i\alpha(x-ct)}, \tag{3.1}$$

where α is the dimensionless wave number ($= 2\pi h/$wavelength), and t is time made non-dimensional by taking h/u_0 as the unit. Either the real or the imaginary part of (3.1) may, of course, be taken to describe a wave on the physical surface. In the usual manner of stability analyses, the wave amplitude δ is taken to be a small quantity whose square is negligible. The corresponding perturbation of the stream function may be written as

$$-\delta f(y) e^{i\alpha(x-ct)},$$

which implies that the velocity components satisfying the continuity condition for incompressible flow are

$$\left.\begin{aligned} u &= \delta f'(y) e^{i\alpha(x-ct)} + U, \\ v &= -i\alpha\delta f(y) e^{i\alpha(x-ct)}. \end{aligned}\right\} \tag{3.2}$$

The elimination of the pressure from the linearized equations of motion leads to an equation for $f(y)$:

$$(U-c)(f''-\alpha^2 f) - U''f = \frac{\nu}{i\alpha u_0 h}(f^{\text{iv}} - 2\alpha^2 f'' + \alpha^4 f), \tag{3.3}$$

which is commonly known as the Orr–Sommerfeld equation (Lin 1955, § 1.3). After substituting for U from (2.1), it is convenient for our purpose to rearrange (3.3) in the form

$$f^{\text{iv}} = (n - cn + 2\alpha^2 - ny^2) f'' + (2n - \alpha^2 n + c\alpha^2 n - \alpha^4 + \alpha^2 n y^2) f, \tag{3.4}$$

and then

$$f^{\text{iv}} = (p + qy^2) f'' + (r + s^2 y^2) f, \tag{3.5}$$

where we have first put $n = i\alpha u_0 h/\nu = 3i\alpha R/2$, and then

$$\left.\begin{aligned} p &= n - cn + 2\alpha^2; & q &= -n; \\ r &= 2n - \alpha^2 n + c\alpha^2 n - \alpha^4; & s^2 &= \alpha^2 n. \end{aligned}\right\} \tag{3.6}$$

This greatly simplifies the writing-out of the subsequent heavy algebra.

The problem entails five boundary conditions as follows. First, the kinematical surface condition is

$$\frac{\partial\eta}{\partial t} + U_0 \frac{\partial\eta}{\partial x} = v_0,$$

where U_0 and v_0 are the values of U and v at $y = 0$; hence, because of (2.1), (3.1) and (3.2),

$$f(0) = c - 1. \tag{3.7}$$

This equation simply shows the connection between the assumed perturbation of the free surface and the corresponding perturbation of the stream function; it bears out the intuitively reasonable expectation that, provided c is not excessively large, the two perturbations are of the same order of magnitude. Since both velocity components must vanish at the fixed boundary, we also have

$$f(1) = 0, \tag{3.8}$$

$$f'(1) = 0. \tag{3.9}$$

The remaining boundary conditions relate to the continuity of stresses across the free surface. It is assumed that the only stress acting on the outside of this surface is a uniform pressure p_0; therefore, since the fluid is viscous, the rate of shearing in the stream $\partial u/\partial y + \partial v/\partial x$ must vanish at $y = \eta$, which requires that

$$f''(0) = 2 - \alpha^2(c-1). \tag{3.10}$$

The equation expressing the continuity of normal stress across the free surface may be written, after division by the density ρ,

$$\frac{p_1}{\rho} - \frac{2\nu u_0}{h}\left(\frac{\partial v}{\partial y}\right)_{y=0} = \frac{p_0}{\rho} + \sigma\Gamma - g\cos\theta\,\eta, \tag{3.11}$$

where p_1 is the (dimensional) pressure just inside the stream, σ is the principal curvature of the surface, and Γ, which may be called the 'kinematic surface tension', is equal to the usual surface tension coefficient divided by density. To the first order in δ, the curvature is

$$\sigma = \frac{1}{h}\frac{\partial^2\eta}{\partial x^2} = -\frac{\alpha^2\delta}{h}e^{i\alpha(x-ct)}. \tag{3.12}$$

Also, the pressure p satisfies the Navier–Stokes equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho u_0^2}\frac{\partial p}{\partial x} + \frac{\nu}{hu_0}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} + hg\sin\theta/u_0^2. \tag{3.13}$$

Hence, with the use of (3.11) and (3.12), separation of the variable part of (3.13) (i.e. the part with δ as a common factor) at $y = \eta$ leads to

$$(n - cn + 3\alpha^2)f'(0) = f'''(0) + \frac{i}{\nu u_0}(\alpha^3\Gamma + \alpha h^2 g\cos\theta). \tag{3.14}$$

The first four of the boundary conditions, (3.7) to (3.10), will be used to determine the four constants arising in the solution of the fourth-order equation (3.4). The boundary condition (3.14) will then be used to find c as a function of α and the parameters of the mean flow. This procedure resembles the usual one in stability problems (cf. Lin 1955). Instability is revealed for a particular wave mode when the respective c turns out to have a positive imaginary part, for then the mode has an exponentially-increasing time factor. As usual it is convenient to express the results of the calculations by setting down the conditions which make c wholly real; c is then simply the phase velocity of a neutral disturbance. These conditions represent a division between stability and instability.

The present problem is somewhat unusual in that two stress conditions, (3.10) and (3.14), replace the two additional kinematic conditions, like (3.8) and (3.9), which arise in problems concerning flow between fixed boundaries. It may, incidentally, be asked why five boundary conditions have been written down whereas most treatments of stability problems (cf. Lin 1955) make do with four? However, the limited meaning of the first boundary condition (3.7) has already been noted above; it contributes nothing to the essential characteristic-value problem, since an assumed perturbation of either the free surface or the stream function may equally well be taken as a starting point, and one simply follows from the other according to (3.7).

The only important difference between this and the better-known problems of hydrodynamic stability lies in the method of solution of the Orr–Sommerfeld equation set out in the next section. In view of existing experimental evidence, attention may confidently be restricted to the case of fairly small Reynolds numbers and wave numbers, and indeed the end results of the calculation confirm the adequacy of this approach.

4. Power-series approximation to the stream function

It was Kelvin (1887) who first noted that in linearized problems like the present one where the function $f(y)$ satisfies the fourth-order differential equation (3.4), the solution may be expressed as an ascending power series in y. He demonstrated that this series is convergent for all physically realizable conditions, but observed that the rate of convergence is very slow when R is not small. The approach suggested by Kelvin is of little use at fairly large Reynolds numbers, with which the greater part of the existing work on hydrodynamic stability is concerned. For the present problem, however, the method is definitely useful, since the values of R and of α are expected to be fairly small in the range of physical interest.

If one formally puts

$$f(y) = \sum_{N=0}^{\infty} A_N y^N, \tag{4.1}$$

then this series is seen to constitute a solution of (3.5) when the coefficients A_N are made to satisfy the recurrence relation

$$N(N-1)(N-2)(N-3)A_N = (N-2)(N-3)pA_{N-2} + \\ + \{r+(N-4)(N-5)q\}A_{N-4} + s^2 A_{N-6} \tag{4.2}$$

for $N > 3$. This relation follows directly from the differential equation. The first four coefficients A_0 to A_3 are simply the four constants arising essentially in the solution of a fourth-order equation, and (4.2) gives every other A_N in terms of these four and the parameters p, q, r, s. If these parameters are now regarded as small quantities of the same order of magnitude, the four coefficients A_4 to A_7 may be said to be $O(p)$, and each successive set of four coefficients is of decreasing order of magnitude with reference to powers of p. From (3.4), (3.5) and (3.6) it is seen that provided α is small, then p, q, r, s are of the order of n: that is, the order of αR. Thus, a limited number of terms of the series would amount to an approximate solution in terms of ascending powers of αR (and, incidentally, of α^2); this may be contrasted to the commonly-used asymptotic solutions in descending powers of αR, suitable when R is large (see, for instance, Lin 1955, chapter 3).

We choose to approximate $f(y)$ in this way as far as $O(p^3) = O(n^3)$, which requires sixteen terms of the expansion (4.1). In this procedure there is an implicit assumption that c is not large, which is confirmed by the final results. In order simply to keep symmetry in the approximation, terms which are $O(\alpha^2)$, for instance, are given equal status with terms which are $O(n)$, although it may be anticipated that $\alpha^2 \ll |n|$ (i.e. $\alpha \ll R$) for the physically interesting cases discussed later.

Although the various manipulations made with the approximate solution for $f(y)$ are very laborious, they are quite straightforward. Accordingly, only an outline of the method of calculation need be given, together with the results of the more important stages.

The third-order approximation to the series solution (4.1), as calculated by means of (4.2), is most conveniently arranged as follows:

$$\begin{aligned}
f(y) = {} & A_0[1 + (1/4!)ry^4 + (1/6!)(pr + s^2)y^6 + (1/8!)(12qr + r^2 + p^2r + 2ps^2)y^8 + \\
& + (1/10!)(2pr^2 + 42pqr + 60qs^2 + 32rs^2)y^{10} + \\
& + (1/12!)(672q^2r + 68qr^2 + r^3)y^{12}] + \\
& + A_1[y + (1/5!)ry^5 + (1/7!)(pr + 6s^2)y^7 + \\
& + (1/9!)(20qr + r^3 + p^2r + 6ps^2)y^9 + \\
& + (1/11!)(62pqr + 2pr^2 + 252qs^2 + 48rs^2)y^{11} + \\
& + (1/13!)(1440q^2r + 92qr^2 + r^3)y^{13}] + \\
& + A_2[y^2 + (2/4!)py^4 + (2/6!)(2q + r + p^2)y^6 + \\
& + (2/8!)(14pq + 2pr + 12s^2 + p^3)y^8 + \\
& + (2/10!)(60q^2 + 32qr + r^2 + 44p^2q + 3p^2r + 42ps^2)y^{10} + \\
& + (2/12!)(560pq^2 + 10pqr + 672qs^2 + 12rs^2)y^{12}] + \\
& + A_3[y^3 + (6/5!)py^5 + (6/7!)(6q + r + p^2)y^7 + \\
& + (6/9!)(26pq + 2pr + 20s^2 + p^3)y^9 + \\
& + (6/11!)(252q^2 + 48qr + r^2 + 68p^2q + 3p^2r + 62ps^2)y^{11} + \\
& + (6/13!)(2124pq^2 + 218pqr + 3pr^2 + 1872qs^2 + 92rs^2)y^{13} + \\
& + (6/15!)(27720q^3 + 5532q^2r + 158qr^2 + r^3)y^{15}].
\end{aligned} \tag{4.3}$$

To use this result, the boundary conditions (3.7), (3.10) and (3.14) are first expressed in terms of the coefficients A_N; these equations give respectively

$$A_0 = c - 1, \tag{4.4}$$

$$2A_2 = 2 - \alpha^2(c-1), \tag{4.5}$$

$$(n - cn + 3\alpha^2)A_1 = 6A_3 + \frac{i}{\nu u_0}(\alpha^3\Gamma + \alpha h^2 g\cos\theta). \tag{4.6}$$

Thus, A_0 and A_2 are related very simply to the main parameters of the problem. To find A_1 and A_3 for use in (4.6), the boundary conditions (3.8) and (3.9) are employed. By setting equal to zero $f(y)$ and $f'(y)$ as given by (4.3), we have a pair of simultaneous equations for A_1 and A_3 in terms of A_0 and A_2. The solutions of these equations are then expressed in ascending powers of p, q, r, s by a straightforward, though rather lengthy, inversion process. By this stage of the calculation, some of the numerical coefficients would become very cumbersome if left as fractions, and so all but the simple coefficients of the leading terms are expressed as decimals. Note that to satisfy (4.6) as far as $O(n^3)$, A_3 is required to $O(n^3)$, but A_1 is required *only to* $O(n^2)$. Accordingly, although A_1 was calculated to $O(n^3)$ to provide certain checks on the result for A_3, the third-order terms in A_1 need not be reproduced here.

After some condensation made possible by the fact that $r = -2q$ in the third-order terms, the results may be expressed as follows:

$$\begin{aligned} A_1 = A_0\Big[& -\tfrac{3}{2} + \left(\tfrac{1}{40}p + \tfrac{1}{140}q + \tfrac{1}{105}r\right) + (-0{\cdot}0013\,095p^2 - \\ & -0{\cdot}0006\,052pq - 0{\cdot}0003\,373pr - 0{\cdot}0000\,773q^2 - \\ & -0{\cdot}0000\,167qr + 0{\cdot}0000\,357r^2 + 0{\cdot}0010\,913s^2)\Big] + \\ + A_2\Big[& -\tfrac{1}{2} + \left(\tfrac{1}{60}p + \tfrac{1}{840}q - \tfrac{1}{840}r\right) + (-0{\cdot}0007\,738p^2 - \\ & -0{\cdot}0001\,389pq + 0{\cdot}0001\,190pr - 0{\cdot}0000\,054q^2 + \\ & +0{\cdot}0000\,123qr - 0{\cdot}0000\,049r^2 - 0{\cdot}0001\,984s^2)\Big], \end{aligned} \tag{4.7}$$

$$\begin{aligned} A_3 = A_0\Big[& \tfrac{1}{2} + \left(-\tfrac{1}{20}p - \tfrac{3}{280}q - \tfrac{11}{280}r\right) + (0{\cdot}0032\,143p^2 + \\ & +0{\cdot}0012\,831pq + 0{\cdot}0010\,450pr + 0{\cdot}0001\,349q^2 + \\ & +0{\cdot}0000\,320qr - 0{\cdot}0000\,891r^2 - 0{\cdot}0022\,487s^2) + \\ & +(-0{\cdot}0001\,759p^3 - 0{\cdot}0000\,315p^2r + 0{\cdot}0000\,130pr^2 + \\ & +0{\cdot}0000\,030r^3 + 0{\cdot}0001\,040ps^2 + 0{\cdot}0000\,507rs^2)\Big] + \\ + A_2\Big[& -\tfrac{1}{2} + \left(-\tfrac{3}{40}p - \tfrac{1}{315}q + \tfrac{1}{315}r\right) + (0{\cdot}0023\,413p^2 + \\ & +0{\cdot}0003\,538pq - 0{\cdot}0003\,108pr + 0{\cdot}0000\,139q^2 - \\ & -0{\cdot}0000\,276qr + 0{\cdot}0000\,119r^2 + 0{\cdot}0003\,638s^2) + \\ & +(-0{\cdot}0001\,068p^3 - 0{\cdot}0000\,758p^2r - 0{\cdot}0000\,192pr^2 - \\ & -0{\cdot}0000\,020r^3 - 0{\cdot}0000\,445ps^2 - 0{\cdot}0000\,122rs^2)\Big]. \end{aligned} \tag{4.8}$$

When p, q, r, s are expressed in terms of n and α^2 by means of (3.6), these results lead to

$$A_1 = A_0\bigg[-\tfrac{3}{2}+\Big(-\frac{1}{40}cn+\frac{31}{840}n+\frac{1}{20}\alpha^2\Big)+ \\ +(-0{\cdot}0013\,095c^2n^2+0{\cdot}0026\,885cn^2-0{\cdot}0012\,798n^2+ \\ +0{\cdot}0147\,619c\alpha^2n-0{\cdot}0138\,095\alpha^2n-0{\cdot}0147\,619\alpha^4)\bigg]+ \\ +A_2\bigg[-\tfrac{1}{2}+\Big(-\frac{1}{60}cn+\frac{11}{840}n+\frac{1}{30}\alpha^2\Big)+ \\ +(-0{\cdot}0007\,738c^2n^2+0{\cdot}0011\,706cn^2-0{\cdot}0004\,463n^2+ \\ +0{\cdot}0019\,048c\alpha^2n-0{\cdot}0013\,492\alpha^2n-0{\cdot}0019\,048\alpha^4)\bigg], \quad (4.9)$$

$$A_3 = A_0\bigg[\tfrac{1}{2}+\Big(\frac{1}{20}cn-\frac{33}{280}n-\frac{1}{10}\alpha^2\Big)+(0{\cdot}0032\,143c^2n^2-0{\cdot}0072\,354cn^2+ \\ +0{\cdot}0037\,357n^2-0{\cdot}0521\,429c\alpha^2n+0{\cdot}0515\,079\alpha^2n+ \\ +0{\cdot}0521\,429\alpha^4)+(0{\cdot}0001\,759c^3n^3-0{\cdot}0004\,963c^2n^3+ \\ +0{\cdot}0004\,519cn^3-0{\cdot}0001\,345n^3-0{\cdot}0032\,712c^2\alpha^2n^2+ \\ +0{\cdot}0054\,363c\alpha^2n^2-0{\cdot}0022\,483\alpha^2n^2+0{\cdot}0083\,413c\alpha^4n- \\ -0{\cdot}0068\,254\alpha^4n-0{\cdot}0065\,926\alpha^6)\bigg]+ \\ +A_2\bigg[-\tfrac{1}{2}+\Big(\frac{3}{40}cn-\frac{11}{168}n-\frac{3}{20}\alpha^2\Big)+(0{\cdot}0023\,413c^2n^2- \\ -0{\cdot}0037\,070cn^2+0{\cdot}0014\,826n^2-0{\cdot}0061\,905c\alpha^2n+ \\ +0{\cdot}0046\,032\alpha^2n+0{\cdot}0061\,905\alpha^4)+(0{\cdot}0001\,068c^3n^3- \\ -0{\cdot}0002\,447c^2n^3+0{\cdot}0001\,881cn^3-0{\cdot}0000\,485n^3- \\ -0{\cdot}0003\,300c^2\alpha^2n^2+0{\cdot}0004\,767c\alpha^2n^2-0{\cdot}0001\,729\alpha^2n^2+ \\ +0{\cdot}0003\,492c\alpha^4n-0{\cdot}0002\,103\alpha^4n-0{\cdot}0002\,328\alpha^6)\bigg]. \quad (4.10)$$

When these expressions are substituted into (4.6) and as much condensation as possible is made, the result is

$$\frac{i}{\nu u_0}(\alpha^3\Gamma+\alpha h^2g\cos\theta)+3c-6+n\Big(-\frac{6}{5}c^2+\frac{68}{35}c-\frac{24}{35}\Big)+\alpha^2(5{\cdot}4c-4{\cdot}8)+ \\ +n^2(-0{\cdot}0057\,143c^3+0{\cdot}0215\,873c^2-0{\cdot}0254\,628c+0{\cdot}0102\,907)+ \\ +\alpha^2n(-0{\cdot}1628\,571c^2+0{\cdot}3029\,095c-0{\cdot}1397\,619)+ \\ +\alpha^4(-0{\cdot}1371\,429c+0{\cdot}0742\,857)+ \\ +n^3(-0{\cdot}0002\,540c^4+0{\cdot}0011\,412c^3-0{\cdot}0018\,007c^2+ \\ +0{\cdot}0012\,414c-0{\cdot}0003\,178)+ \\ +\alpha^2n^2(0{\cdot}0073\,968c^3-0{\cdot}0240\,506c^2+0{\cdot}0257\,008c-0{\cdot}0090\,556)+ \\ +\alpha^4n(-0{\cdot}0506\,667c^2+0{\cdot}0966\,906c-0{\cdot}0468\,574)+ \\ +\alpha^6(0{\cdot}0547\,302c-0{\cdot}0504\,127) = 0. \quad (4.11)$$

This result can be regarded as an equation for c in terms of n and α, hence of R and α, and of the 'restoring force' due to gravity and surface tension as expressed by the first group of terms in the equation. In particular, the equation can be used to find the relation between R and α which makes c wholly real, i.e. which gives the case of neutral stability. To do this, we first separate the real part of (4.11), noting that n is purely imaginary, and hence quite easily obtain an explicit expression for c by a process of successive approximation in terms of α^2 and $n^2 = -9\alpha^2R^2/4$. We finally get, for c real,

$$c = 2(1-\alpha^2+11\alpha^4/6+0{\cdot}0077\,581\alpha^2R^2-3{\cdot}3555\,556\alpha^6), \qquad (4.12)$$

where the values of R and α must be consistent with the imaginary part of (4.11) being satisfied. The error in (4.12) is known to be considerably less, due to numerical factors, than $O(\alpha^4R^4)$. Note that there is no term in $(\alpha R)^1$ in (4.12); this result occurs by the cancellation of terms with fairly complicated numerical factors. (This may seem perhaps to suggest some special significance; but I can see no other way of arriving at such a result beyond a direct calculation like the present one.)

The imaginary part of (4.11) is now considered. On eliminating c by means of (4.12), and using the following relations found easily from (2.2) and (2.3):

$$u_0^2 h = (3^{5/3}/4)(g\sin\theta)^{1/3}\nu^{4/3}R^{5/3},$$
$$u_0^2/h = \tfrac{3}{4}g\sin\theta R,$$

we finally obtain the approximation

$$\frac{4\zeta}{3^{5/3}}\frac{\alpha^2}{R^{5/3}}+\frac{4\cot\theta}{3R}-\frac{8}{5}+5{\cdot}5289\,142\alpha^2-$$
$$-0{\cdot}0000\,639\alpha^2R^2-14{\cdot}6352\,952\alpha^4 = 0, \qquad (4.13)$$

where ζ is defined by

$$\zeta = \Gamma(g\sin\theta)^{-1/3}\nu^{-4/3}. \qquad (4.14)$$

Equation (4.13) expresses the required approximate relation between R and α for neutral stability. The equation is easily solved, since for a specified value of R it becomes a quadratic in α^2.

In passing it may be noted that the formula for real values of c, i.e. (4.12) subject to (4.11), appears to differ from the result obtained by Yih (1954), which was presented only in graphical form. The discrepancy is not understood. For $\alpha \to 0$ (i.e. for very long waves), equation (4.12) gives simply $c = 2$, which agrees with the result (2.4) obtained by a very simple argument. This also agrees with the result found by Kapitza (1948), who used a different form of argument applicable only to very long waves.

Let us now consider, for example, the case of a vertical wall ($\theta = 90°$). The term involving $\cot\theta$ in (4.13) now vanishes, and we have $\zeta = \Gamma g^{-1/3}\nu^{-4/3}$. (The parameter ζ is about 3300 for water at 19° C.) Figure 2 shows the relation between R and α according to (4.13) for various values of ζ and a range of R up to 20. The curves shown in the figure are curves of neutral stability; and it can be inferred that, for a particular ζ, the region lying above the respective curve represents stability, while that below it represents instability. The following features of figure 2 may be specially noted.

(i) Every curve for $\zeta > 0$ passes through the origin, and has infinite slope there. (The case $\zeta = 0$ is discussed below.) Thus, for all finite values of R there is a finite range of unstable α, and therefore the flow is never completely stable. On the other hand, equation (4.13) shows that stability occurs for sufficiently small R when $0 < \theta < 90°$.

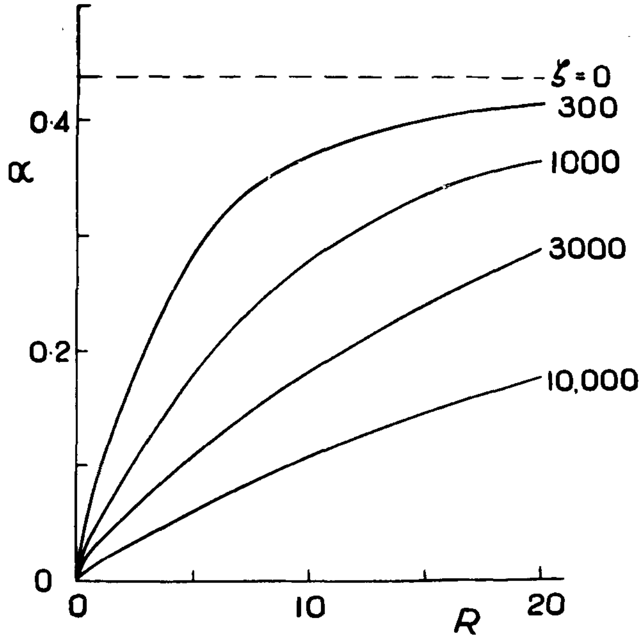

Figure 2. Curves of neutral stability for laminar flow down a vertical wall with various values of the parameter $\zeta = \Gamma\, g^{-1/3}\, \nu^{-4/3}$.

(ii) The stabilizing tendency of surface tension is demonstrated by the fact that the curves bend nearer the R-axis for increasing ζ. Thus, as might be expected intuitively, the range of unstable α for a given R is reduced as Γ, and hence ζ, is increased. Note that surface tension cannot induce complete stability, although it may stabilize waves whose length is less than a certain limit.

(iii) The curve according to (4.13) for $\zeta = 0$ is included in the figure, being shown as a broken line, although the values of α in this case are rather too large for confidence in the accuracy of the result. (We recall that the theory has been developed on the assumption that α is fairly small.) However, the trend of the other curves in the figure shows definitely that the unstable region of the (R, α)-chart is largest when $\zeta = 0$. This fact is useful, since it seems to rule out the possibility that the neglect of surface tension in Yih's treatment of the problem may be responsible for the discrepancy between his and the present general conclusions regarding stability.

In connection with the general role of surface tension in the mathematical problem, the following matter is worth noting. Equation (4.13) shows that $\alpha \to O(R^{5/6}/\zeta^{1/2})$ for $R \to 0$ with $\theta = 90^\circ$. Thus, for small R the curve α *vs* $R^{5/3}$ is a parabola whose vertex touches the α-axis at the origin. The curvature at the vertex is directly proportional to ζ, and so in the limit $\zeta \to 0$ the curvature is zero and the curve coincides with the α-axis. It may hence be inferred that the curve α *vs* R also coincides with the α-axis for $\zeta \to 0$, although **for** this curve the curvature at the origin is infinite for for all finite values of ζ. Clearly, the same result should occur if surface tension were ignored (i.e. $\zeta = 0$) from the outset of the analysis, and so one might question why no such result is forthcoming if we put directly $\zeta = 0$ and $\theta = 90^\circ$ in (4.13). However, in proceeding from the dimensional form of the dynamical equation (3.13) to as far as (4.13), a division of all the terms by $u_0^2 h$, which is proportional to $R^{5/3}$, has been entailed. Hence, to preserve the appropriate 'physical' status of surface tension with respect to the limiting process $R \to 0$, equation (4.13) should be multiplied by $R^{5/3}$; when this is done there is no ambiguity about the fact that $R = 0$ is a solution regardless of α when $\zeta = 0$. This result that the flow is unstable, or at least neutral, for all finite R and α seems correct intuitively, since for a vertical wall and zero surface tension there is no restoring force to act on the disturbed flow.

(iv) Owing to the particular method of approximation used for the theory, the most accurate part of the figure is the vicinity of the origin. The upper half, at least, of the figure is somewhat uncertain; but the figure as a whole is quite adequate as a qualitative picture of the neutral-stability conditions. Moreover, the line of argument to be developed in § 5 shows that it is sufficient to consider a range of R and α very near the origin to account satisfactorily for the relevant experimental facts.

5. Simplified theory for long waves

In this section some formulae are derived which are suitable for a straightforward check with experiment. In contrast with the results of the last section, they are compact and easily manipulated; and although these advantages are gained by sacrificing the higher-order terms calculated in § 4, the loss of precision entailed is found not to be serious. Let us therefore proceed on the assumption, to be justified later, that α is quite small. Our knowledge of the second- and third-order terms in (4.11) shows conclusively that a valid first approximation to this equation is simply

$$\frac{i}{\nu u_0}\{\alpha^3\Gamma + \alpha h^2 g \cos\theta\} + 3c - 6 + \frac{3}{2} i\alpha R\left\{-\frac{6}{5}c^2 + \frac{68}{35}c - \frac{24}{35}\right\} = 0. \quad (5.1)$$

Suppose now that $c = c_r + ic_i$, where c_r and c_i are wholly real, and that $c_r \gg c_i$. The latter assumption will also be justified later. On separation of real and imaginary parts, equation (5.1) gives, as a first approximation,

$$c_r = 2, \quad (5.2)$$

and hence

$$c_i = \frac{1}{2} R \left\{ \frac{8}{5} \alpha - \frac{\alpha^3 \Gamma}{h u_0^2} - \frac{\alpha h g \cos\theta}{u_0^2} \right\}. \tag{5.3}$$

Therefore, since unstable waves have $c_i > 0$, the condition for instability is

$$\frac{8}{5} > \frac{\alpha^2 \Gamma}{h u_0^2} + \frac{h g \cos\theta}{u_0^2}. \tag{5.4}$$

Some special cases following from (5.4) deserve to be noted. If one puts $\alpha \to 0$ in (5.4) and substitutes (2.2) and (2.3) in the second term on the right-hand side, this instability condition becomes

$$R > \frac{5}{6} \cot\theta. \tag{5.5}$$

Thus, when (5.5) is just satisfied, very long waves ($\alpha \to 0$) become unstable. Moreover, if surface tension is zero, all fairly long waves are unstable when (5.5) is satisfied, which rather suggests that in practice a bore (in other words, a discontinuous 'roll wave') would be formed. It is notable, though not at all unexpected, that except for the numerical factor the instability condition (5.5) is identical with the condition found by Jeffreys (1925) for the formation of roll waves on a turbulent stream, assumed to be subject to the Chézy law of friction. Indeed, the present result can be deduced directly, with only a very small discrepancy in the numerical factor, from the roll-wave theory developed by Dressler (1952) for a generalized law of friction. The details of this comparison need not be given here; it is sufficient to note that the result follows in a straightforward way when the primary-flow relationships for laminar flow (as in § 2) are introduced into Dressler's generalized instability condition.

The action of gravity may be expected to encourage instability if $\theta > 90°$, that is, if the fixed wall slopes backwards and the stream runs down its under side. The ability of liquid streams to adhere beneath solid surfaces is popularly known as the 'tea-pot effect', and will be familiar to any one who has poured liquid from a vessel with an ill-designed spout. The destabilizing influence of gravity in this case is properly expressed by the gravitational term in (5.4), which becomes negative when $\theta > 90°$. As an extreme case, let $\theta = 180°$ and hence $u_0 = 0$ in accordance with (2.2). The liquid now forms a stationary film on the under side of a horizontal plane, and the instability condition (5.4) becomes

$$g h^2 > \alpha^2 \Gamma,$$

or

$$\lambda > 2\pi (\Gamma/g)^{1/2}, \tag{5.6}$$

if the wavelength $\lambda = 2\pi h/\alpha$ is introduced. This is seen to check with the result obtained by Bellman & Pennington (1954, equation (3.7)), who calculated the effect of surface tension on the Taylor instability of the boundary between two fluids accelerated perpendicularly to its plane

(Taylor 1950). (It should be noted that the effect of gravity is here equivalent to an upward acceleration.)

The final special case which we shall discuss arises when $\theta = 90^\circ$, that is, when the fixed boundary is vertical. This is the case for which a comparison between theory and experiment will be made in § 6, and also the one in which there is marked disagreement between the present work and that of Yih (1954). The rest of this section will be devoted to it.

For $\theta = 90^\circ$, (5.4) becomes

$$\frac{8}{5} > \frac{\alpha^2 \Gamma}{h u_0^2}. \tag{5.7}$$

Hence, the flow is always unstable, since sufficiently small values of α can always be found to satisfy this instability condition; indeed, this fact is already clear from the discussion at the end of § 4 and also from (5.5), which becomes simply $R > 0$ when $\theta = 90^\circ$. As pointed out in an early part of this paper, this result is remarkable in view of existing experimental work on vertical liquid films, much of which seems to indicate an absence of waves at sufficiently low Reynolds numbers. However, the following straightforward argument appears to bring the present theory into a completely satisfactory agreement with experiment, in particular the work mentioned in § 6.

It is reasonable to suppose that the wave most prominent in an experimental observation will be the one whose rate of growth according to linearized theory is largest. The use of such an assumption has many precedents; for instance, Rayleigh (1894, § 87) discussed its theoretical basis, and used it in several of his investigations. There is various experimental evidence to suggest that often the growth of a 'wave of maximum instability' eventually, through non-linear effects, tends to suppress other waves which are unstable according to linearized theory but have smaller rates of growth; thus, the well-defined pattern of the particular wave mode of greatest rate of amplification may be observed (see Benjamin & Ursell 1954 for a discussion of this point in a rather similar context). In the present problem unstable waves grow in amplitude according to a time factor $\exp(\alpha c_i t)$. To find α for the most unstable wave, we have therefore only to maximize αc_i, which, by virtue of (5.3), amounts to maximizing the function

$$\frac{8}{5}\alpha^2 - \frac{\alpha^4 \Gamma}{h u_0^2}.$$

Hence, the optimum value of α (say α_m) is given by

$$\alpha_m^2 = \frac{4 h u_0^2}{5 \Gamma}. \tag{5.8}$$

When the formulae of § 2 for the primary flow are used (with $\theta = 90^\circ$) to eliminate u_0 and h from this expression, it leads directly to

$$\alpha_m = 1{\cdot}12\{\nu^{2/3} g^{1/6} \Gamma^{-1/2}\} R^{5/6}, \tag{5.9}$$

where the numerical factor is an approximation to $(3/5)^{1/2}3^{1/3}$. This together with (5.3) shows that the maximum value of c_i is

$$(c_i)_m = 0{\cdot}224\{\nu^{2/3}g^{1/6}\Gamma^{-1/2}\}R^{11/6}. \tag{5.10}$$

The quantity within the braces in (5.9) and (5.10) depends only on the particular liquid considered. For example, taking the values appropriate to water at 19° C: $\nu = 0{\cdot}0103$ cm²/sec and $\Gamma = 72{\cdot}9$ cm³/sec², and also $g = 981$ cm/sec², we get from (5.9) and (5.10)

$$\alpha_m = 0{\cdot}0195R^{5/6}, \tag{5.11}$$

$$(c_i)_m = 0{\cdot}00391R^{11/6}. \tag{5.12}$$

Provided R is not much greater than unity, these results justify the initial assumptions that α and c_i are small.

Now, to illustrate the likelihood of waves being observed experimentally, we shall calculate the amplification experienced by the wave of maximum instability on a vertical water film as it travels a distance of 10 cm. Since the wave velocity is approximately $2u_0$, the amplification factor is approximately $\mathscr{A} = \exp\{10\alpha_m(c_i)_m/2h\}$, where h is in cm. Hence, using (5.11) and (5.12), and eliminating h by means of the primary-flow relations of § 2, we finally get

$$\mathscr{A} = \exp\{0{\cdot}0543R^{8/3}\}. \tag{5.13}$$

Values of $\mathscr{A}$ given by (5.13) are plotted as a function of R in figure 3. The figure shows the amplification to be large only for values of R greater than about 4; above this the rate of increase of $\mathscr{A}$ is enormous. Thus, one might expect observable waves to develop reasonably near the beginning of the stream as soon as the Reynolds number is brought up to a value near 4. Of course, the linearized theory ceases to apply when large amplifications occur; but the present result does seem to offer a convincing explanation of why, in experiments where the Reynolds number is gradually increased from small values, waves should suddenly appear on a previously untroubled stream. However, it has been shown here that a critical Reynolds number in the usual sense does not exist for a vertical stream; for, even when the Reynolds number is extremely small, there are unstable waves which, presumably, would grow to appreciable size if the stream were long enough. The present result suggests that experimental measurements of the 'quasi-critical' Reynolds number (R_{qc} say, which seems from the theory to be about 4 for water) may depend to a large extent on the method and precision of the observations (even perhaps on the eyesight of the observer); this may explain the wide scatter among existing estimates. Note that the value of R_{qc} depends in a fairly complicated way on the kinematic surface tension and viscosity; according to the present theory, it is definitely not the same for different liquids.

In conclusion a formula for the dimensional wavelength λ_m corresponding to the most unstable wave number α_m is worth noting, being more convenient than (5.9) for comparison with experiment. Putting $\lambda_m = 2\pi h/\alpha_m$ and using (2.3) and (5.9), we find that

$$\lambda_m = 8{\cdot}11\{\Gamma^{1/2}g^{-1/2}\}R^{-1/2}. \tag{5.14}$$

For water at 19° C, this gives

$$\lambda_m = 2{\cdot}21R^{-1/2}\ \text{cm} = 0{\cdot}871R^{-1/2}\ \text{in.} \tag{5.15}$$

This is suggested as the most likely wavelength of observed disturbances, at least when instability first becomes noticeable, and a wave structure can be distinguished.

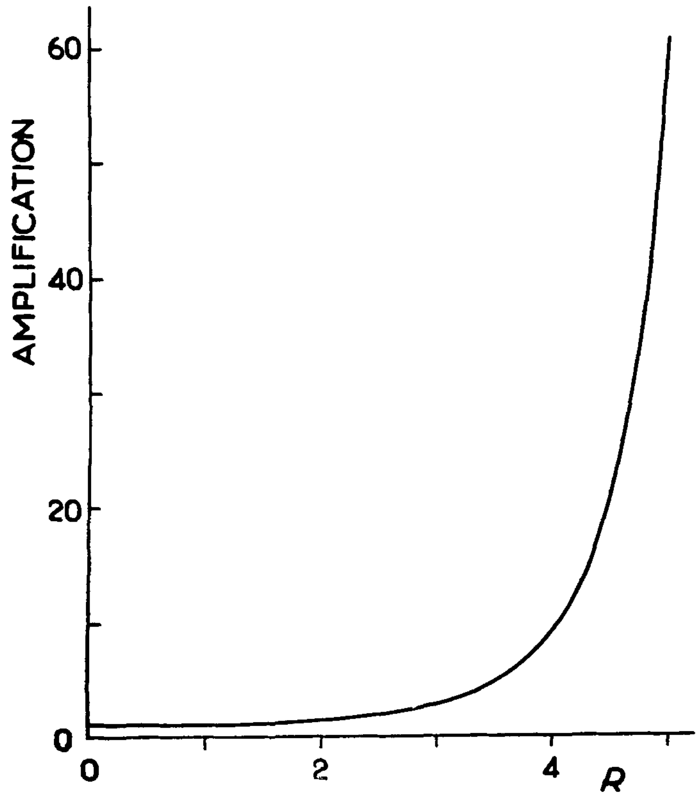

Figure 3. The amplification experienced by the most unstable wave on a vertical water film as the wave travels 10 cm.

Nevertheless, one can scarcely expect waves to appear with a strictly uniform and distinct periodicity; because under all conditions infinitesimal waves with a wide range of wavelengths are unstable, and the wave with length λ_m comes into prominence only through a rather uncritical selection process depending on differences in the rates of amplification at different wavelengths. The ultimate state of the amplified waves is, of course, determined largely by non-linear effects which remain unknown.

6. Comparison with the experiments by Binnie

The reader may refer to the paper by Binnie (1957) for an account of these experiments, which were carried out with water at 19° C. The waves described by him were somewhat irregular in length even when the flow was delicately adjusted until the waves were just perceptible (see the photograph in his paper). This feature is understandable from the considerations made in the final paragraph of § 5, and suggests the unlikelihood of a very precise agreement between experiment and any linearized theory.

The three main points of comparison are as follows:

(i) The Reynolds number at which observable waves first occur was estimated experimentally as 4·4. Although the theory does not provide a correspondingly precise estimate, this value appears quite reasonable in the light of equation (5.13) and figure 3.

(ii) The records of wave velocity and length were taken when the measured value of Q was $6{\cdot}9 \times 10^{-3}$ in.^{3}/sec per inch span. On the substitution of this value and the value of ν quoted in § 5 for water at 19° C, equations (2.2) and (2.3) (with $\theta = 90°$) lead to $h = 4{\cdot}4 \times 10^{-3}$ in. and hence $2u_0 = 4{\cdot}7$ in./sec. According to (5.2) the latter figure is the first approximation to the wave velocity. The mean experimental value for the wave velocity was 5·5 in./sec, i.e. 17% in excess of the theoretical estimate. The order of magnitude of the extra terms involved in an approximation to c_r better than (5.2) is found to be too small to account for this discrepancy, which seems more likely to be due to the finite size of the measured waves. (The velocities of other kinds of surface wave are known to increase with the wave amplitude; e.g. irrotational waves in an inviscid fluid (Lamb 1932, § 250).) However, the comparison seems quite good in view of the obvious limitations of linearized theory in this type of investigation; in any case, it appears definitely to support the present theoretical results in favour of those found by Yih (1954) which indicate that the wave velocity should be many times greater than the experimental values.

(iii) With $R = 4{\cdot}4$ as measured, equation (5.15) gives $\lambda_m = 0{\cdot}42$ in. for the most unstable wavelength. This is in surprisingly close agreement with the value 0·45 in. obtained as the mean of the experimental observations.

The above three items seem in large measure to confirm the usefulness of the simplified method developed in § 5. However, further experiments would be necessary for a complete check; for instance, measurements of R_{qc} for different liquids would provide a particularly severe test. Suitable experiments on films with a finite slope would be more difficult than those on vertical films, since the convenience of a cylindrical wall is lost and difficulties with edge effects are bound to arise, but such experiments would be very desirable.

I wish to acknowledge the encouragement given throughout this work by Mr A. M. Binnie, and his invaluable help in checking the lengthy algebraic calculations of § 4. I should like also to express my thanks to Professor T. R. C. Fox, who first stimulated my interest in this problem, and to Dr G. K. Batchelor, who noticed that a powerful criticism based on energy considerations could be brought against my results for a vertical stream; this proved indecisive in the end, but led to a helpful clarification of some aspects of the problem.

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Note added at proof stage

In the course of a recent correspondence with Prof. C.-S. Yih, light has been thrown on the possible reasons for the discrepancy between the present results and those published by him in 1954, to which frequent reference has been made in the text above. He has also kindly communicated some valuable comments on general aspects of the stability problem. It seems desirable that the main points to have emerged from the correspondence should be put on record here, particularly since this may obviate the confusing position which the differences between our two papers may otherwise have created. The points listed for convenience as follows are all due essentially to Prof. Yih, although the wording is chiefly mine.

(1) The direct numerical method used by Yih (1954) in calculating the neutral stability conditions for a vertical film was based on only two terms of a certain series expansion, and was rather sensitive to small computational errors. Thus, although his results were an effective demonstration of the low Reynolds numbers at which instability can be expected, they cannot be relied upon as more than a rough indication of the range of

unstable conditions ; in particular, they cannot be held to confute the present precise results concerning instability near the origin of the (R, α)-diagram.

(2) In unpublished work, Yih has obtained a proof that, for a vertical film, disturbances with sufficiently large wave number are stable irrespectively of surface tension or Reynolds number. This conclusion is perfectly in accord with the present work, although beyond its scope since the nature of the approximations introduced here preclude an accurate account of very short waves.

(3) The case of zero wave number (i.e. indefinitely long waves) is somewhat ambiguous from the mathematical point of view. For the long-wave approximation used in this paper, the wave number is taken to be small enough for its square to be negligible but remains a finite quantity throughout the calculations. This approach, which seems the most reasonable physically, reveals instability in a vertical film even for vanishingly small wave numbers. On the other hand, the long-wave case might be approached by setting the wave number equal to zero from the start. This step corresponds to a strictly uni-directional disturbance, with the free surface entirely undisturbed and undisplaced. In fact, the disturbance need not be infinitesimal, since the Navier–Stokes equations can be exactly solved. Yih has observed that the latter approach leads to a result indicating stability. This result appears to indicate that there is a higher mode which is always damped at zero wave number.

(4) The distinctive features of this problem, notably the instability of a vertical film at all Reynolds numbers, all stem from the fact that the energy supply to the flow is from gravity potential—in contrast to, say, the energy supply from the free stream in the case of boundary layers. If gravity were absent, any motion of the film would eventually be damped out ; in this sense the film would be stable, although, of course, any form of *static* disturbance is neutral when surface tension is also absent.

(5) From the mathematical point of view, a distinctive feature of the problem is that the complex wave velocity c appears not only in the differential equations but also in the boundary conditions. This fact, in addition to those presented in the last three items, points to the possibility that for each pair of values of R and α there may be two values of c—one for the primary mode and one for the higher mode. There are no numerical results for this possible higher mode, which in any case is not as significant physically as the primary mode.